

CONVOLUTIONS ON THE HAAGERUP TENSOR PRODUCTS OF FOURIER ALGEBRAS

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ABSTRACT. We study the ranges of the maps of convolution $u \otimes v \mapsto u * v$ and a ‘twisted’ convolution $u \otimes v \mapsto u * \check{v}$ ($\check{u}(s) = u(s^{-1})$) and on the Haagerup tensor product of a Fourier algebra of a compact group $A(G)$ with itself. We compare the results to result of factoring these maps through projective and operator projective tensor products. We notice that $(A(G), *)$ is an operator algebra and observe an unexpected set of spectral synthesis.

1. INTRODUCTION

Let G be a compact group and $A(G)$ be its Fourier algebra, in the sense of [10].

In [12], questions of the following nature were addressed: what are the ranges of convolution and ‘twisted’ convolution, when applied to $A(G \times G) = A(G) \hat{\otimes} A(G)$ (operator projective tensor product). The authors’ motivation was two-fold. First, these particular maps played a fundamental role in the famous result of B. Johnson ([16]) that $A(G)$ is sometimes non-amenable, and the authors were interested in seeing how these techniques related to the completely bounded theory of Fourier algebras. This perspective led the authors to the results of [13]. Secondly, it was observed is that ‘twisted’ convolution averages $A(G \times G)$ over left cosets of the diagonal subgroup $\Delta = \{(s, s) : s \in G\}$, whereas convolution averages $A(G \times G)$ over orbits of the group action $(r, (s, t)) \mapsto (sr^{-1}, rt) : G \times (G \times G) \rightarrow G \times G$. Thus the images may be rightly regarded as Fourier algebras of certain homogeneous/orbit spaces of $G \times G$ in the sense of [11]. The homogeneous space $G \times G/\Delta$ and the orbit space $G \times G/\check{\Delta}$ may be naturally identified with G . Thus we define $\Gamma, \check{\Gamma} : \mathcal{C}(G \times G) \rightarrow \mathcal{C}(G)$ by

$$\Gamma u(s) = \int_G u(sr, r) dr \text{ and } \check{\Gamma} u(s) = \int_G u(sr, r^{-1}) dr.$$

It is easy to check that $\Gamma(u \otimes v) = u * v$ and $\check{\Gamma}(u \otimes v) = u * \check{v}$.

In [9], it was shown that the Haagerup tensor product $A(G) \otimes^h A(G)$ is a Banach algebra. By [22, Thm. 2] this algebra has spectrum $G \times G$. We shall note, below,

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that $A(G) \otimes^h A(G)$ is, in fact, semi-simple, and may thus be regarded as an algebra of functions on $G \times G$. Hence it is natural to ask whether we discover anything new if we apply the maps Γ and $\check{\Gamma}$ to $A(G) \otimes^h A(G)$. While we gain no new spaces, we learn interesting comparisons between $A(G) \otimes^h A(G)$, $A(G \times G)$ and $\mathcal{C}(G) \otimes^h \mathcal{C}(G)$. See §3, below.

1.1. Some basic results. We let for each π in \widehat{G} , $\text{Trig}_\pi = \text{span}\{s \mapsto \langle \pi(s)\xi | \eta \rangle : \xi, \eta \in \mathcal{H}_\pi\}$ and $\text{Trig}(G) = \bigoplus_{\pi \in \widehat{G}} \text{Trig}_\pi$. This has linear dual space $\text{Trig}(G)' = \prod_{\pi \in \widehat{G}} \mathcal{B}(\mathcal{H}_\pi)$ via dual pairing

$$(1.1) \quad \langle u, T \rangle = \sum_{\pi \in \widehat{G}} d_\pi \text{Tr}(\hat{u}(\pi) T_\pi)$$

where $\hat{u}(\pi) = \int_G u(s) \pi(s^{-1}) ds$. If $T \in \text{Trig}(G)'$, we let \check{T} be defined by $\langle u, \check{T} \rangle = \langle \check{u}, T \rangle$ in the duality (1.1). Here $\check{u}(s) = u(s^{-1})$. The following is surely well-known, and recorded here for later convenience.

Lemma 1.1. *For $T \in \text{Trig}(G)'$, then for π in \widehat{G} , $\check{T}_\pi = T_\pi^\dagger$. Here the conjugation and transpose are realised with respect to the same orthonormal basis for \mathcal{H}_π .*

Proof. We first compute that

$$\hat{\check{u}}(\pi) = \int_G u(s) \pi(s) ds = \int_G u(s) \bar{\pi}(s^{-1})^\dagger ds = \hat{u}(\bar{\pi})^\dagger.$$

Hence $\text{Tr}(\hat{\check{u}}(\pi) T_\pi) = \text{Tr}(\hat{u}(\bar{\pi}) T_\pi^\dagger)$. Thus in (1.1) we simply change π to $\bar{\pi}$. \square

We will identify the left regular representation up to quasi-equivalence as

$$\lambda = \bigoplus_{\pi \in \widehat{G}} \pi \quad \text{on} \quad \mathcal{H} = \ell^2\text{-}\bigoplus_{\pi \in \widehat{G}} \mathcal{H}_\pi.$$

It is standard that $\lambda(G)$ is weak*-dense in $\text{Trig}(G)'$ in terms of the duality (1.1). The von Neumann algebra generated by $\lambda(G)$ is thus the operator space direct product

$$\text{VN}(G) = \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} \mathcal{B}(\mathcal{H}_\pi).$$

Observe that this algebra has centre $\text{ZVN}(G) = \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} \mathbb{C} I_\pi$. The following is also surely well-known. Again, we provide a proof for convenience of the reader.

Proposition 1.2. *For each π in \widehat{G} , we have that $A \mapsto \int_G \pi(s^{-1}) A \pi(s) ds : \mathcal{B}(\mathcal{H}_\pi) \rightarrow \mathbb{C} I_\pi$ is a tracial expectation; hence $\int_G \pi(s^{-1}) A \pi(s) ds = \frac{1}{d_\pi} \text{Tr}(A) I_\pi$. Thus $T \mapsto \int_G \lambda(s^{-1}) T \lambda(s) ds : \text{VN}(G) \rightarrow \text{ZVN}(G)$ is a tracial expectation given by*

$$\int_G \lambda(s^{-1}) T \lambda(s) ds = \bigoplus_{\pi \in \widehat{G}} \frac{1}{d_\pi} \text{Tr}(T_\pi) I_\pi.$$

Proof. It is easy to see that $\int_G \pi(s^{-1}) A \pi(s) ds$ commutes with each element of $\pi(G)$, hence by Schur's lemma is a scalar operator, and preserves I_π . Hence, $\int_G \pi(s^{-1}) A \pi(s) ds = \omega(A) I_\pi$ for some functional ω . Likewise

$$\begin{aligned} \int_G \pi(s^{-1}) A \pi(t) \pi(s) ds &= \int_G \pi(s^{-1}) A \pi(ts) ds \\ &= \int_G \pi(s^{-1}t) A \pi(s) ds = \int_G \pi(s^{-1}) \pi(t) A \pi(s) ds \end{aligned}$$

and as $\text{span } \pi(G) = \mathcal{B}(\mathcal{H}_\pi)$ we see that $\int_G \pi(s^{-1})AB\pi(s) ds = \int_G \pi(s^{-1})BA\pi(s) ds$. By uniqueness of the normalised trace, $\omega = \frac{1}{d_\pi} \text{Tr}$. The second result is immediate. \square

The Fourier algebra is the predual of $\text{VN}(G)$ via the dual pairing (1.1). Hence we obtain complete isometric identification

$$(1.2) \quad A(G) = \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi S_{d_\pi}^1$$

where S_d^1 denotes the $d \times d$ -matrices with trace norm; i.e. for u in $A(G)$ we have

$$\|u\|_A = \sum_{\pi \in \widehat{G}} d_\pi \|\hat{u}(\pi)\|_{S^1}$$

where $\|\hat{u}(\pi)\|_{S^1}$ is the trace-norm of the $d_\pi \times d_\pi$ -matrix $\hat{u}(\pi)$.

An *operator space structure* on a given complex linear space \mathcal{X} is an assignment of norms, one on each space $M_n(\mathcal{X})$ for natural number n , which satisfies certain compatibility conditions; see, for example M1 and M2 of [8, p. 20]. We shall not require these explicit axioms here. Of importance to us, are the following facts. First, any von Neuman algebra \mathcal{V} , in particular $\text{VN}(G)$, will have assigned to each $M_n(\mathcal{V})$ the unique norm which realises it as a von Neumann algebra. Maps between operator spaces, $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ are those maps whose matrix amplifications $\Phi^{(n)} : M_n(\mathcal{X}) \rightarrow M_n(\mathcal{Y})$, $\Phi^{(n)}[X_{ij}] = [\Phi(X_{ij})]$ are uniformly bounded: $\|\Phi\|_{cb} = \sup_n \|\Phi^{(n)}\| < \infty$. The space $\mathcal{CB}(\mathcal{X}, \mathcal{Y})$ of completely bounded maps is itself an operator space via the isometric identifications $[\Phi_{ij}] \mapsto (X \mapsto [\Phi_{ij}(X)]) : M_n(\mathcal{CB}(\mathcal{X}, \mathcal{Y})) \rightarrow \mathcal{CB}(\mathcal{X}, M_n(\mathcal{Y}))$, where $M_m(M_n(\mathcal{X})) = M_{mn}(\mathcal{X})$. In particular linear functionals are automatically completely bounded, and $\mathcal{X}^* = \mathcal{CB}(\mathcal{X}, \mathbb{C})$ inherits the operator space structure prescribed above. If \mathcal{V} is a von Neumann algebra, then its predual \mathcal{V}_* inherits the operator space structure from the inclusion $\mathcal{V}_* \hookrightarrow \mathcal{V}^*$. A map $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ is called a complete isometry of each $\Phi^{(n)}$ is an isometry, and a complete quotient if each $\Phi^{(n)}$ is a quotient map. In the latter case we say \mathcal{Y} is a complete quotient space of \mathcal{X} . See [8, I.3] for details.

Now if \mathcal{A} is any Banach space of functions on G for which $\text{Trig}(G)$ is dense in \mathcal{A} , then \mathcal{A}^* may be realised as a subspace of $\text{Trig}(G)'$. Generally \mathcal{A} will be a subspace of $A(G)$ (these will arise from application of Γ and $\tilde{\Gamma}$; see [12, Prop. 1.3]), whence $\mathcal{V} = \text{VN}(G)$ will be frequently used as below.

Given an operator space \mathcal{A} , a subspace \mathcal{V} of \mathcal{A}^* is *completely norming* if for each n and $[x_{ij}]$ in $M_n(\mathcal{A})$ we have

$$\|[x_{ij}]\|_{M_n(\mathcal{A})} = \sup \left\{ \|[f_{kl}(x_{ij})]\|_{M_{mn}} : [f_{kl}] \in M_m(\mathcal{V}), \|[f_{kl}]\|_{M_m(\mathcal{A}^*)} \leq 1, m \in \mathbb{N} \right\}.$$

If \mathcal{V} is any weak*-dense subspace of \mathcal{A}^* , then it is completely norming. Indeed, this follows from the fact that the embedding of \mathcal{A} into \mathcal{A}^{**} is a complete isometry [8, 3.2.1], and then that $M_n(\mathcal{V})$ is weak*-dense in $M_n(\mathcal{A}^*)$, where the latter is identified as $\mathcal{CB}(\mathcal{A}, M_n) \cong (\mathcal{A} \widehat{\otimes} S_n^1)^*$ [8, (7.1.6)].

Lemma 1.3. *Let \mathcal{X} and \mathcal{Y} be operator spaces, $\Lambda : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear map with dense range, and \mathcal{V} be a weak*-dense subspace of \mathcal{Y}^* . Then $\mathcal{A} = \Lambda(\mathcal{X})$, assigned the operator space structure in such a manner that it is regarded as a complete quotient of \mathcal{X} , admits \mathcal{V} as a weak*-dense subspace of its dual \mathcal{A}^* . In*

particular, when \mathcal{V} is given the operator space structure by which $\Lambda^* : \mathcal{V} \rightarrow \mathcal{X}^*$ is a complete isometry, then \mathcal{V} is completely norming for \mathcal{A} .

Proof. The density of \mathcal{A} in \mathcal{Y} allows us to view \mathcal{V} as a subspace of \mathcal{A}^* . Furthermore, if a in \mathcal{A} satisfies $f(a) = 0$ for each f in \mathcal{V} , then $f(a) = 0$ for each f in \mathcal{Y}^* , hence $a = 0$. Thus $\{f \in \mathcal{V} : f|_{\mathcal{A}} = 0\} = \{0\}$, and bipolar theorem allows us to conclude that \mathcal{V} is weak*-dense in \mathcal{Y}^* . Moreover, Λ^* is a complete isometry exactly when Λ is a complete quotient map, thanks to [8, 4.1.9]. Hence we see that \mathcal{V} is completely norming, thanks to remarks in the last paragraph, above. \square

1.2. The Haagerup tensor product of Fourier algebras. Fix a Hilbert space \mathcal{H} . In [19, 3] it is shown that each weak*-weak* continuous completely bounded operator Φ on $\mathcal{B}(\mathcal{H})$ — we shall write $\Phi \in \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ — is given by

$$(1.3) \quad \Phi(T) = \sum_{i \in I} V_i T W_i$$

where $\{V_i, W_i\}_{i \in I}$ is a family in $\mathcal{B}(\mathcal{H})$ for which each of the series $\sum_{i \in I} V_i V_i^*$ and $\sum_{i \in I} W_i^* W_i$ is weak*-convergent. We shall write $\Phi = \sum_{i \in I} V_i \otimes W_i$, accordingly. Furthermore, we have completely bounded norm

$$\|\Phi\|_{cb} = \min \left\{ \left\| \sum_{i \in I} V_i V_i^* \right\|^{1/2} \left\| \sum_{i \in I} W_i^* W_i \right\|^{1/2} : \Phi = \sum_{i \in I} V_i \otimes W_i \text{ as in (1.3)} \right\}$$

and operator composition

$$(1.4) \quad \sum_{i \in I} V_i \otimes W_i \circ \sum_{i' \in I} V_{i'}' \otimes W_{i'}' = \sum_{i \in I} \sum_{i' \in I} V_i V_{i'}' \otimes W_{i'}' W_i.$$

Hence it is sensible to write

$$\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}) \otimes^{w^*h} \mathcal{B}(\mathcal{H})$$

and call this space the *weak* Haagerup tensor product*. This is known to be the same as the extended Haagerup tensor product of [9]; see, for example, the treatment of [20, §2].

Let $\mathcal{V} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and \mathcal{V}' its commutant. It is shown in [19, 3] that subspace of \mathcal{V}' -bimodule maps in $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ are exactly those element which admit a representation as in (1.3) with each V_i, W_i in \mathcal{V} . The description of the norm, with minimum taken over elements V_i and W_i form \mathcal{V} , and the operator composition are maintained, making this a closed subalgebra of $\mathcal{B}(\mathcal{H}) \otimes^{w^*h} \mathcal{B}(\mathcal{H})$. We denote this space by $\mathcal{V} \otimes^{w^*h} \mathcal{V}$. Let \mathcal{V}_* denote the predual of \mathcal{V} . We define for an elementary tensor $u = v \otimes w$ in $\mathcal{V}_* \otimes \mathcal{V}_*$ and $\Phi = \sum_{i \in I} V_i \otimes W_i$ in $\mathcal{V} \otimes^{w^*h} \mathcal{V}$, the dual pairing

$$\langle u, \Phi \rangle = \sum_{i \in I} \langle v, V_i \rangle \langle w, W_i \rangle.$$

and define the Haagerup norm on $\mathcal{V}_* \otimes \mathcal{V}_*$ by

$$\|u\|_h = \sup\{|\langle u, \Phi \rangle| : \Phi \in \mathcal{V} \otimes^{w^*h} \mathcal{V}, \|\Phi\|_{cb} \leq 1\}.$$

We then let $\mathcal{V}_* \otimes^h \mathcal{V}_*$ denote the completion of $\mathcal{V}_* \otimes \mathcal{V}_*$ with respect to this norm. Then, as shown in [3] the dual pairing above extends to a duality

$$(1.5) \quad (\mathcal{V}_* \otimes^h \mathcal{V}_*)^* \cong \mathcal{V} \otimes^{w^*h} \mathcal{V}.$$

Then $\mathcal{V}_* \otimes^h \mathcal{V}_*$ gains its operator space structure as being a distinguished predual of $\mathcal{V} \otimes^{w^*h} \mathcal{V} \subset \mathcal{CB}^\sigma(\mathcal{B}(H))$, as in [8, (3.2.2) & (3.2.11)]. [The more traditional method of defining $\mathcal{V}_* \otimes^h \mathcal{V}_*$ is to assign \mathcal{V}_* the predual operator space structure and use tensor formulas such as in [8, II.9]. This gives an equivalent but non-intuitive description of $\mathcal{V}_* \otimes^h \mathcal{V}_*$. We will make extensive use only of (1.5), above.]

Let us return to the Fourier algebra $A(G)$ of a compact group. Recall, as in the last section, that $A(G)$ is the predual of the von Neumann algebra $VN(G) \subset \mathcal{B}(\mathcal{H})$ where $\mathcal{H} = \ell^2\text{-}\bigoplus_{\pi \in \widehat{G}} \mathcal{H}_\pi$. Then we define the Haagerup tensor product of $A(G)$ with itself in terms of the completely isometric duality, as in (1.5), above

$$(1.6) \quad (A(G) \otimes^h A(G))^* \cong VN(G) \otimes^{w^*h} VN(G) \subset \mathcal{CB}(\mathcal{B}(\mathcal{H})).$$

Let us note a result promised above.

Proposition 1.4. *The Banach algebra $A(G) \otimes^h A(G)$ is semi-simple.*

Proof. We observe that $VN(G) \otimes VN(G)$ is weak*-dense in $VN(G) \otimes^{w^*h} VN(G) \cong (A(G) \otimes^h A(G))^*$, by [3, Thms. 3.1 & 3.2]. It is obvious that $\text{span } \lambda(G) \otimes \lambda(G)$ is weak*-dense in $VN(G) \otimes VN(G)$, where $\lambda : G \rightarrow VN(G)$ identifies G with the spectrum of $A(G)$. Thus the bipolar theorem tells us that $(\text{span } \lambda(G) \otimes \lambda(G))^\perp = \{0\}$, whence $\lambda(G) \otimes \lambda(G)$ separates points in $A(G) \otimes^h A(G)$. \square

2. MAIN RESULTS

2.1. ‘Twisted’ convolution. Our main method for dealing with understanding Γ , as defined on either of $A(G) \hat{\otimes} A(G)$, or on $A(G) \otimes^h A(G)$, is to study its adjoint. To this end consider $\Gamma : \text{Trig}(G) \otimes \text{Trig}(G) \rightarrow \text{Trig}(G)$. If $u, v \in \text{Trig}(G)$ and $t \in G$ we have

$$\langle \Gamma(u \otimes v), \lambda(t) \rangle = \int_G u(s) v(t^{-1}s) ds = \int_G \langle u \otimes v, \lambda(s) \otimes \lambda(t^{-1}) \lambda(s) \rangle ds$$

and hence we have that $\Gamma^*(\lambda(t)) = \int_G \lambda(s) \otimes \lambda(t^{-1}) \lambda(s) ds$, in a weak* sense. By weak*-density of $\text{span } \lambda(G)$ in $\text{Trig}(G)'$ we conclude that for T in $\text{Trig}(G)'$ we have

$$(2.1) \quad \Gamma^*(T) = \int_G \lambda(s) \otimes \check{T} \lambda(s) ds$$

where the integral is understood in the weak* sense and \check{T} is defined as in Lemma 1.1.

We define $A_\Delta(G) = \Gamma(A(G) \hat{\otimes} A(G))$. In [12] this was regarded as a quotient space of $A(G \times G) \cong A(G) \hat{\otimes} A(G)$ and assigned a norm accordingly. We augment the concrete realisation of this norm, computed in [12, Thm. 2.2], by specifying the operator space structure on $A_\Delta(G)$ in a concrete manner.

Theorem 2.1. *The operator space structure on $A_\Delta(G)$, qua complete quotient of $A(G) \hat{\otimes} A(G)$ by Γ , is given by the weighted operator space direct sum*

$$A_\Delta(G) = \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{3/2} S_{d_\pi, r}^2$$

where $S_{d, r}^2$ denotes the $d \times d$ matrices with Hilbert-Schmidt norm and row operator space structure.

Proof. We recall that $(A(G) \hat{\otimes} A(G))^* \cong VN(G) \bar{\otimes} VN(G)$ by [7] (see [8, 7.2.4]). We consider the adjoint $\Gamma^* : VN(G) \rightarrow VN(G) \bar{\otimes} VN(G)$ with n th amplification $\Gamma^{*(n)} : M_n(VN(G)) \rightarrow VN(G) \bar{\otimes} M_n(VN(G))$. We have from (2.1) that

$$\Gamma^{*(n)}[T_{ij}] = \int_G \lambda(t) \otimes [\check{T}_{ij}] \lambda(t)^{(n)} ds$$

and hence

$$\begin{aligned} \Gamma^{*(n)}[T_{ij}]^* \Gamma^{*(n)}[T_{ij}] &= \int_G \int_G \lambda(s^{-1}t) \otimes \lambda(s^{-1})^{(n)} [\check{T}_{ij}]^* [\check{T}_{ij}] \lambda(t)^{(n)} ds dt \\ &= \int_G \int_G \lambda(s^{-1}) \otimes \lambda(s^{-1})^{(n)} \lambda(t^{-1})^{(n)} \sum_{k=1}^n [\check{T}_{ki}^* \check{T}_{kj}] \lambda(t)^{(n)} ds dt \\ &= \int_G \lambda(s) \otimes \lambda(s)^{(n)} ds \cdot \sum_{k=1}^n I \otimes \int_G [\lambda(t^{-1}) \check{T}_{ki}^* \check{T}_{kj} \lambda(t)] dt. \end{aligned}$$

We observe that

$$P_n = \int_G \lambda(s) \otimes \lambda(s)^{(n)} ds$$

is evidently a self-adjoint projection. Furthermore, the Schur orthogonality relations tell us that on $\mathcal{H} \otimes^2 \mathcal{H}^n = \ell^2\text{-}\bigoplus_{\pi', \pi \in \hat{G}} \mathcal{H}_{\pi'} \otimes^2 \mathcal{H}_{\pi}^n$ we have

$$P_n = \bigoplus_{\pi \in \hat{G}} \int_G \bar{\pi}(s) \otimes \pi(s)^{(n)} ds$$

and hence this projection is non-zero only in each anti-diagonal component of $VN(G) \bar{\otimes} M_n(VN(G)) = \ell^\infty\text{-}\bigoplus_{\pi', \pi \in \hat{G} \times \hat{G}} \mathcal{B}(\mathcal{H}_{\pi'}) \bar{\otimes} M_n(\mathcal{B}(\mathcal{H}_{\pi}))$. However, by Proposition 1.2 we see that

$$\sum_{k=1}^n I \otimes \int_G [\lambda(t^{-1}) \check{T}_{ki}^* \check{T}_{kj} \lambda(t)] dt = \sum_{k=1}^n \bigoplus_{\pi \in \hat{G}} \frac{1}{d_\pi} I \otimes [\text{Tr}(\check{T}_{ki, \pi}^* \check{T}_{kj, \pi}) I_\pi]$$

Thus we see that

$$\begin{aligned} \|\Gamma^{*(n)}[T_{ij}]\| &= \|\Gamma^{*(n)}[T_{ij}]^* \Gamma^{*(n)}[T_{ij}]\|^{1/2} \\ &= \sup_{\pi \in \hat{G}} \frac{1}{d_\pi^{1/2}} \left\| \sum_{k=1}^n [\text{Tr}(\check{T}_{ki, \pi}^* \check{T}_{kj, \pi})] \right\|^{1/2} \\ &= \sup_{\pi \in \hat{G}} \frac{1}{d_\pi^{1/2}} \left\| \sum_{k=1}^n [\text{Tr}(T_{ki, \bar{\pi}}^* T_{kj, \bar{\pi}})] \right\|^{1/2} \end{aligned}$$

where we have appealed to Lemma 1.1 in the last line. According to [8, (3.4.4)] we obtain completely isometric embedding

$$\Gamma^*(VN(G)) \subseteq \ell^\infty\text{-}\bigoplus_{\pi \in \hat{G}} \frac{1}{d_\pi^{1/2}} S_{d_\pi, c}^2$$

Thus Lemma 1.3 and the duality of operator Hilbert spaces [8, (3.4.4)] provide the desired result. We observe, moreover, that our dual pairing (1.1) is bilinear, which is why we need not concern ourselves with conjugate spaces. \square

The operator space structure of Theorem 2.1 provides alternate explanation for curious results discovered in the some prior articles.

Corollary 2.2. (i) [12, Prop. 2.5] *We have $A_\Delta(G) \hat{\otimes} A_\Delta(G) \cong A_\Delta(G \times G)$.*
(ii) [13, Prop. 4.3] *The map $u \mapsto \check{u}$ is a complete isometry on $A_\Delta(G)$.*

Proof. Consider the identifications

$$\begin{aligned} \left(\ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi S_{d_\pi, r}^2 \right) \hat{\otimes} \left(\ell^1\text{-}\bigoplus_{\pi' \in \widehat{G}} d_{\pi'} S_{d_{\pi'}, r}^2 \right) &\cong \ell^1\text{-}\bigoplus_{\pi, \pi' \in \widehat{G} \times \widehat{G}} d_\pi d_{\pi'} S_{d_\pi, r}^2 \hat{\otimes} S_{d_{\pi'}, r}^2 \\ &\cong \ell^1\text{-}\bigoplus_{\pi, \pi' \in \widehat{G} \times \widehat{G}} d_\pi d_{\pi'} S_{d_\pi d_{\pi'}, r}^2 \end{aligned}$$

where we have appealed to the fact that the tensor product of row spaces is again a row space ([8, 9.3.5]). Since $\widehat{G \times G} \cong \widehat{G} \times \widehat{G}$, (i) follows.

The map $u \mapsto \check{u} : \text{Trig}_\pi \rightarrow \text{Trig}_{\bar{\pi}}$, which is an isometry from $d_\pi S_{d_\pi}^2$ to $d_{\bar{\pi}} S_{d_{\bar{\pi}}}^2$ – practically the transpose – is a complete isometry with row structure, thanks to [8, 3.4.2]. Thus by the structure of the direct product, we get (ii). \square

The next result is a bit of a surprise. It shows that $A(G) \otimes^h A(G)$ behaves exactly as does $A(G) \hat{\otimes} A(G)$ with respect to Γ .

Theorem 2.3. *We have that $\Gamma(A(G) \otimes^h A(G)) = A_\Delta(G)$. Moreover, if $A_\Delta(G)$ is given the operator space structure in Theorem 2.1, above, then $\Gamma : A(G) \otimes^h A(G) \rightarrow A_\Delta(G)$ is a complete quotient map.*

Proof. Let $A_\Delta^h(G) = \Gamma(A(G) \otimes^h A(G))$, and assign it the operator space structure which makes Γ a complete quotient map. The completely contractive inclusion $A(G) \hat{\otimes} A(G) \hookrightarrow A(G) \otimes^h A(G)$ gives, via the fact that $A_\Delta(G)$ is a complete quotient of $A(G) \hat{\otimes} A(G)$, a completely contractive inclusion $A_\Delta(G) \hookrightarrow A_\Delta^h(G)$. Since $\text{Trig}(G)$ is dense in both subspaces, $A_\Delta(G)$ is dense in $A_\Delta^h(G)$, so there the adjoint $A_\Delta^h(G)^* \hookrightarrow A_\Delta(G)^*$ is completely contractive and injective. We wish to see that this map is a complete isometry. It suffices to appeal to Lemma 1.3 and verify that on $M_n(\text{VN}(G))$, we have

$$(2.2) \quad \|[T_{ij}]\|_{M_n(A_\Delta^h(G)^*)} \leq \|[T_{ij}]\|_{M_n(A_\Delta(G)^*)}.$$

We recall the duality relation (1.6). Hence for $[T_{ij}]$ in $M_n(\text{VN}(G))$ we have by (2.1) and the last identification immediately above

$$\begin{aligned} \|[T_{ij}]\|_{M_n(A_\Delta^h(G)^*)} &= \left\| \Gamma^{*(n)}[T_{ij}] \right\|_{CB(\mathcal{B}(\mathcal{H}), M_n(\mathcal{B}(\mathcal{H})))} \\ &= \left\| A \mapsto \int_G [\lambda(s) A \check{T}_{ij} \lambda(s)] ds \right\|_{CB(\mathcal{B}(\mathcal{H}), M_n(\mathcal{B}(\mathcal{H})))} \\ &= \sup \left\{ \left| \int_G \langle [\lambda(s) A_{kl} \check{T}_{ij} \lambda(s)] \xi | \eta \rangle ds \right| : \begin{array}{l} [A_{kl}] \in \text{ball}(M_m(\mathcal{B}(\mathcal{H}))) \\ \xi, \eta \in \text{ball}(\mathcal{H}^{mn}), m \in \mathbb{N} \end{array} \right\}. \end{aligned}$$

We observe for operator matrix $[A_{kl}]$ and vectors ξ, η as above that

$$\begin{aligned}
& \left| \int_G \langle [\lambda(s)A_{kl}\check{T}_{ij}\lambda(s)]\xi | \eta \rangle ds \right| \\
& \leq \int_G \| [A_{kl}\check{T}_{ij}\lambda(s)]\xi \| \left\| \lambda(s^{-1})^{(nm)}\eta \right\| ds \\
& \leq \left(\int_G \| [A_{kl}\check{T}_{ij}\lambda(s)]\xi \|^2 \right)^{1/2} \left(\int_G \left\| \lambda(s^{-1})^{(nm)}\eta \right\|^2 ds \right)^{1/2} \\
& \leq \left(\int_G \langle [A_{kl}\check{T}_{ij}\lambda(s)]^* [A_{kl}\check{T}_{ij}\lambda(s)]\xi | \xi \rangle \right)^{1/2} \\
& \leq \left\langle \int_G \sum_{k=1}^n [\lambda(s^{-1})\check{T}_{ki}^* \check{T}_{kj}\lambda(s)]\xi | \xi \right\rangle^{1/2} \\
& \leq \left\| \int_G \sum_{k=1}^n [\lambda(s^{-1})\check{T}_{ki}^* \check{T}_{kj}\lambda(s)] \right\|^{1/2}.
\end{aligned}$$

However, calculations in the proof of Theorem 2.1 reveal that the last quantity is exactly

$$\sup_{\pi \in \widehat{G}} \frac{1}{d_\pi^{1/2}} \left\| \sum_{k=1}^n [\text{Tr}(T_{ki,\pi}^* T_{kj,\pi})] \right\|^{1/2} = \| [T_{ij}] \|_{M_n(A_\Delta(G)^*)}.$$

Thus (2.2) is established. \square

2.2. Convolution. We now wish to consider the map $\check{\Gamma}$ on $A(G) \otimes^h A(G)$. Consider $\check{\Gamma} : \text{Trig}(G) \otimes \text{Trig}(G) \rightarrow \text{Trig}(G)$. If $u, v \in \text{Trig}(G)$ and $t \in G$ we have

$$\langle \check{\Gamma}(u \otimes v), \lambda(t) \rangle = \int_G u(s)v(s^{-1}t) ds = \int_G \langle u \otimes v, \lambda(s) \otimes \lambda(s^{-1})\lambda(t) \rangle ds$$

and hence we have that $\Gamma^*(\lambda(t)) = \int_G \lambda(s) \otimes \lambda(s^{-1})\lambda(t) ds$, in a weak* sense. By weak*-density of $\text{span } \lambda(G)$ in $\text{Trig}(G)'$ we conclude that for T in $\text{Trig}(G)'$ we have

$$(2.3) \quad \check{\Gamma}^*(T) = \int_G \lambda(s) \otimes \lambda(s^{-1})T ds$$

where the integral is understood in the weak* sense.

Theorem 2.4. *We have $\check{\Gamma}(A(G) \otimes^h A(G)) = A(G)$. Moreover, $\check{\Gamma} : A(G) \otimes^h A(G) \rightarrow A(G)$ is a complete quotient map.*

Proof. Let $T \in \text{VN}(G)$. Then $\check{\Gamma}^*(T)$, via the identification (1.6) and using the composition product of (1.4), factors as

$$\begin{aligned}
\check{\Gamma}^*(T) &= \int_G \lambda(s) \otimes \lambda(s^{-1})T ds = (I \otimes T) \circ \int_G \lambda(s) \otimes \lambda(s^{-1}) ds \\
&\text{i.e. } \check{\Gamma}^*(T)(A) = \left(\int_G \lambda(s)A\lambda(s^{-1}) ds \right) T.
\end{aligned}$$

Hence if $[T_{ij}] \in M_n(VN(G))$ then, since the map from Proposition 1.2 is an expectation, hence completely contractive, we have

$$\begin{aligned} & \left\| \check{\Gamma}^{*(n)}[T_{ij}] \right\|_{\mathcal{CB}(\mathcal{B}(\mathcal{H}), M_n(\mathcal{B}(\mathcal{H})))} \\ & \leq \|A \mapsto [AT_{ij}]\|_{\mathcal{CB}(\mathcal{B}(\mathcal{H}), M_n(\mathcal{B}(\mathcal{H})))} \left\| A \mapsto \int_G \lambda(s) A \lambda(s^{-1}) ds \right\|_{\mathcal{CB}(\mathcal{B}(\mathcal{H}))} \\ & \leq \|[T_{ij}]\|_{M_n(VN(G))}. \end{aligned}$$

Conversely, inspecting this operator at $A = I$ we observe

$$\left\| \check{\Gamma}^{*(n)}[T_{ij}] \right\|_{\mathcal{CB}(\mathcal{B}(\mathcal{H}), M_n(\mathcal{B}(\mathcal{H})))} \geq \|[T_{ij}]\|_{M_n(VN(G))}.$$

Thus equality holds.

Since $\check{\Gamma}(A(G) \otimes^h A(G))$ contains $\text{Trig}(G)$ which is dense in $A(G)$, and since $\check{\Gamma}^*$ is a complete isometry, the desired results hold. \square

We obtain an immediate corollary of Theorem 2.4, by using the result [1] (see [2, 5.2.1]) that any algebra and operator space \mathcal{A} whose product completely boundedly factors through $\mathcal{A} \otimes^h \mathcal{A}$ is completely isomorphic to an operator algebra, i.e., an algebra of operators acting on a Hilbert space.

Corollary 2.5. *The convolution algebra $(A(G), *) = (A(G), \check{\Gamma})$ is completely isomorphic to an operator algebra.*

In fact, by [2, 5.2.8], we see that the representation $\rho : (A(G), *) \rightarrow \mathcal{B}(\mathcal{K})$ can be set to satisfy $\|\rho\|_{\text{cb}} \leq 2$ and $\|\rho^{-1}\|_{\text{cb}} \leq 1$ (the latter on $\rho(A(G))$). Since $(A(G), *)$ is a Segal algebra in $L^1(G)$, it does not admit a bounded approximate identity ([4, Theo. 1.2]), in particular it does not admit a contractive approximate identity. Hence we do not have enough information to establish if ρ can be made a complete isometry (as would follow from the Blecher-Ruan-Sinclair theorem as stated in [2, 2.3.2], for example).

This stands in mild contrast to [2, 5.5.8], where it is shown that the ‘matrix’ algebra S_∞^1 is not an operator algebra. Of course, the global structure of $(A(G), *)$ is much different, since convolution on $\text{Trig}_\pi \otimes \text{Trig}_\pi \cong d_\pi S_{d_\pi}^1 \otimes d_\pi S_{d_\pi}^1$ is really matrix multiplication times a scalar factor $\frac{1}{d_\pi}$ (see [15, (27.20)]); hence we obtain a contraction $d_\pi S_{d_\pi}^1 \otimes^h d_\pi S_{d_\pi}^1 \rightarrow d_\pi S_{d_\pi}^1$.

Let us close this section with a remark on convolution applied to $A(G \times G)$. The methods are very close to those of [12, Thm. 4.1], but we make the extra effort to gain the operator space structure.

Proposition 2.6. *We have a completely isometric identification*

$$\check{\Gamma}(A(G \times G)) = \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^2 S_{d_\pi}^1$$

where we regard this space as a complete quotient space of $A(G \times G)$ by $\check{\Gamma}$.

We remark that the space above was denoted $A_\gamma(G)$ in [16]. In [17, Def. 2.6] $A_\gamma(G)$ is regarded as a Beurling-Fourier algebra, and is given the same operator space structure, though in terms of a certain weighted dual pairing with $VN(G)$, which is a different perspective than the one taken here.

Proof. As above, we need only determine the norm of $\check{\Gamma}^{*(n)}[T_{ij}]$ in $M_n(VN(G) \bar{\otimes} VN(G)) \cong M_n(VN(G \times G))$, for $[T_{ij}]$ in $M_n(VN(G))$. Using (2.3) and factoring in the product in $M_n(VN(G) \bar{\otimes} VN(G))$ we have

$$\begin{aligned} \check{\Gamma}^{*(n)}[T_{ij}] &= \left(\int_G \lambda(s) \otimes \lambda(s^{-1}) ds \right)^{(n)} [I \otimes T_{ij}] \\ &= \bigoplus_{\pi \in \widehat{G}} \left(\int_G \pi(s) \otimes \pi(s^{-1}) ds \right)^{(n)} [I \otimes T_{ij, \pi}]. \end{aligned}$$

It is straightforward to calculate that each $\int_G \pi(s) \otimes \pi(s^{-1}) ds = \frac{1}{d_\pi} U_\pi$ where U_π is a unitary, in fact a permutation matrix. Thus we obtain a completely isometric embedding

$$\check{\Gamma}^*(VN(G)) \subseteq \ell^\infty - \bigoplus_{\pi \in \widehat{G}} \frac{1}{d_\pi} S_{d_\pi}^\infty$$

and the desired result follows. \square

3. COMPARISON OF RESULTS

We let $\mathcal{C}(G)$ denote the space of continuous functions on G . The *Varopoulos algebra* is given by

$$V(G \times G) = \mathcal{C}(G) \otimes^\gamma \mathcal{C}(G) = \mathcal{C}(G) \hat{\otimes} \mathcal{C}(G) = \mathcal{C}(G) \otimes^h \mathcal{C}(G)$$

where isomorphic equality of the spaces is provided by Grothendieck's inequality [6, 14.5]. We shall take \otimes^h to define our canonical norm on $V(G \times G)$. Since the map $u \mapsto \check{u}$ is a complete isometry on $\mathcal{C}(G)$, we have that $\check{\Gamma}(V(G \times G)) = \Gamma(V(G \times G))$ completely isometrically. We recall that $A(G) \hat{\otimes} A(G) = A(G \times G)$ completely isometrically, by virtue of the facts that $(A(G) \hat{\otimes} A(G))^* = VN(G) \bar{\otimes} VN(G)$, as indicated in the proof of Theorem 2.1, and the latter space is $VN(G \times G)$; while $A(G) \otimes^\gamma A(G) = A(G \times G)$ isomorphically only when G has an abelian subgroup of finite index [18], and isometrically only when G is abelian. The fact that $u \mapsto \check{u}$ is an isometry on $A(G)$ means that $\Gamma(A(G) \otimes^\gamma A(G)) = \check{\Gamma}(A(G) \otimes^\gamma A(G))$.

algebra	image under Γ	image under $\check{\Gamma}$	references
$V(G \times G)$	$A(G)$	$A(G)$	[21]
$A(G) \otimes^h A(G)$	$A_\Delta(G)$	$A(G)$	§2.1, §2.2
$A(G \times G)$	$A_\Delta(G)$	$A_\gamma(G)$	[12] (§2.1, §2.2)
$A(G) \otimes^\gamma A(G)$	$A_\gamma(G)$	$A_\gamma(G)$	[16]

Notice that in each of the first three rows, Γ and $\check{\Gamma}$ can be regarded as a complete quotient map, as shown in §2.1 and §2.2, above.

Let us use Theorems 2.3 and 2.4 to observe some further connections between $A(G \times G)$ and $A(G) \otimes^h A(G)$, and also between $V(G \times G)$ and $A(G) \otimes^h A(G)$. This addresses a question asked in [5, p. 21]. We use the same definitions as in [21, 12]. The equivalence of (i), (ii) and (iii) below, is an immediate consequence of [12, Theo. 1.4]. The equivalence of (i'), (ii') and (iii') is proved essentially as [21, Thm. 3.1]; see [13, Lem. 2.3]. We refer the reader to those sources for the proof.

Proposition 3.1. *Let $\theta, \check{\theta} : G \times G \rightarrow G$ be given by $\theta(s, t) = st^{-1}$ and $\check{\theta}(s, t) = st$. Also, let $E \subset G$ be closed. Then the following are equivalent:*

- (i) *E is a set of spectral synthesis for $A_\Delta(G)$;*
- (ii) *$\theta^{-1}(E)$ is a set of spectral synthesis for $A(G \times G)$;*

(iii) $\theta^{-1}(E)$ is a set of spectral synthesis for $A(G) \otimes^h A(G)$.

Also, the following are equivalent:

(i') E is a set of spectral synthesis for $A(G)$;

(ii') $\check{\theta}^{-1}(E)$ is a set of spectral synthesis for $V(G \times G)$;

(iii') $\check{\theta}^{-1}(E)$ is a set of spectral synthesis for $A(G) \otimes^h A(G)$.

It is well-known, see for example [14], that point sets are spectral for $A(G)$. Since $\check{\theta}^{-1}(\{e\}) = \{(s, s^{-1}) : s \in G\}$, we gain the following.

Corollary 3.2. *The anti-diagonal $\check{\Delta} = \{(s, s^{-1}) : s \in G\}$ is a set of spectral synthesis for $A(G) \otimes^h A(G)$.*

This stands in marked contrast to the case for $A(G \times G)$: $\check{\Delta}$ is a set of spectral synthesis for $A(G \times G)$ if and only if the connected component of the identity G_e is abelian ([13, Thm. 2.5]).

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REFERENCES

- [1] D.P. Blecher. A completely bounded characterization of operator algebras. *Math. Ann.* 303 (1995), no. 2, 227–239.
- [2] D.P. Blecher and C. Le Merdy. *Operator algebras and their modules – an operator space approach.* volume 30 of *London Math. Soc., New Series*. Clarendon Press, Oxford, New York, 2004.
- [3] D.P. Blecher and R.R. Smith. The dual of the Haagerup tensor product. *J. London Math. Soc. (2)*, 45 (1992), no. 1, 126–144.
- [4] J.T. Burnham. Closed ideals in subalgebras of Banach algebras. I. *Proc. Amer. Math. Soc.* 32 (1972), 551–555.
- [5] M. Daws. Multipliers and abstract harmonic analysis. Talk at *Canadian Abstract harmonic Analysis Symposium*, Saskatoon, 2010.
See http://www1.maths.leeds.ac.uk/~mdaws/talks/cahas_hand.pdf.
- [6] A. Defant and K. Floret. *Tensor norms and operator ideals*, volume 163 of *North-Holland Mathematics Studies*. North-Holland, Amsterdam, 1993.
- [7] E.G. Effros and Z.-J. Ruan. On approximation properties for operator spaces. *Internat. J. Math.* 1 (1990), no. 2, 163–187.
- [8] E.G. Effros and Z.-J. Ruan. *Operator Spaces*, volume 23 of *London Math. Soc., New Series*. Clarendon Press, Oxford, 2000.
- [9] E.G. Effros and Z.-J. Ruan. Operator space tensor products and Hopf convolution algebras. *J. Operator Theory*, 50 (2003), no. 1, 131–156.
- [10] P. Eymard. L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France*, 92 (1964), 181–236.
- [11] B.E. Forrest. Fourier analysis on coset spaces. *Rocky Mountain J. Math.* 28 (1998), no. 1, 173–190.
- [12] B.E. Forrest, E. Samei and N. Spronk. Convolutions on compact groups and Fourier algebras of coset spaces. *Studia Math.* 196 (2010), no. 3, 223–249.
- [13] B.E. Forrest, E. Samei and N. Spronk. Weak amenability of Fourier algebras on compact groups. *Indiana Univ. Math. J.* 58 (2009), no. 3, 1379–1393.
- [14] C. Herz. Harmonic synthesis for subgroups. *Ann. Inst. Fourier (Grenoble)* 23 (1973), no. 3, 91–123.
- [15] E. Hewitt and K.A. Ross. *Abstract harmonic analysis. Vol. II.*, volume 152 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York-Berlin, 1970.
- [16] B.E. Johnson. Non-amenability of the Fourier algebra of a compact group. *J. London Math. Soc. (2)*, 50 (1994), no. 2, 361–374.
- [17] H.H. Lee and E. Samei. Beurling-Fourier algebras, operator amenability and Arens regularity. *J. Funct. Anal.*, 262 (2012), no. 1, 167–209.

- [18] V. Losert. On tensor products of Fourier algebras. *Arch. Math. (Basel)*, 43 (1984), no. 4, 370–372.
- [19] R.R. Smith. Completely bounded module maps and the Haagerup tensor product. *J. Funct. Anal.*, 102 (1991), no. 1, 156–175.
- [20] N. Spronk. Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras. *Proc. London Math. Soc. (3)*, 89 (2004), no. 1, 161–192.
- [21] N. Spronk and L. Turowska. Spectral synthesis and operator synthesis for compact groups. *J. London Math. Soc. (2)*, 66 (2002), no. 2, 361–376.
- [22] J. Tomiyama. Tensor products of commutative Banach algebras. *Tôhoku Math. J. (2)*, 12 (1960), 147–154.